Convergence Analysis of a Discontinuous Finite Element Formulation based on Second Order Derivatives

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Abstract

A new Discontinuous Galerkin Formulation is introduced for the elliptic reaction-diffusion problem that incorporates local second order distributional derivatives. The corresponding bilinear form satisfies both coercivity and continuity properties on the broken Hilbert space of H^2 functions. For piecewise polynomial approximations of degree $p \geq 2$, optimal uniform h and p convergence rates are obtained in the broken H^1 and H^2 norms. Convergence in L^2 is optimal for $p \geq 3$, if the computational mesh is *strictly rectangular*. If the mesh consists of skewed elements, then optimal convergence is only obtained if the corner angles satisfy a given regularity condition. For $p = 2$, only suboptimal h convergence rates in L^2 are obtained and for linear polynomial approximations the method does not converge.

Key words: Discontinuous Galerkin Methods, *a priori* error estimates.

1 Introduction

Several variations of Discontinuous Galerkin Methods (DGM's) for second order elliptic boundary value problems have been proposed during recent years, which exhibit special convergence, conservation and local approximation properties attractive for parallel adaptive hp-approximations. A comprehensive account of several types of DGM's can be found in the volume edited by Cockburn, Karniadakis and Shu [1], in the paper of Arnold, Brezzi, Cockburn, and Marini [2], and in the report by Prudhomme et al. [3].

In 1997, Oden, Babuška, and Baumann [4] introduced a discontinuous Galerkin formulation similar to the GEM formulation by Delves et al. [5,6], but sign differences in certain terms resulted in a positive definite bilinear form. For $p \geq 2$ (where p denotes the minimum order of the polynomial approximations),

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the DGM by Oden, Babuška, and Baumann appears to be unconditionally stable, whereas the GEM formulation requires the inclusion of penalty terms to stabilize the formulation. Complete details are given in [7].

Rivière et al. [8–10] proposed an extension of the DGM of Oden, Babuška, and Baumann by including a penalty term on the jumps of the solution across the element interfaces. The formulation, due to the addition of a penalty term, does not satisfy a local conservation property any longer, but it becomes stable for $p = 1$. Moreover, optimal h and suboptimal p convergence rates are proved in the broken H^1 space.

In this paper, a new DGM formulation is presented for a model class of linear elliptic boundary value problems, that incorporates the second order distributional derivatives, and consequently is defined within the space of local H^2 functions. The formulation does not require any penalization or stabilization, exhibits a local conservation property, and *guarantees optimal uniform h and p convergence in the broken H^2 and H^1 spaces* when the order of polynomial approximation is at least of degree ≥ 2. This is an improvement over other DGM formulations for which optimal h convergence has been proved but optimal p convergence rates have not been established.

This formulation exhibits eccentric convergence behavior in L^2. When the computational meshes employ solely rectangular elements, it is proved (and numerically confirmed) that optimal h and p convergence rates hold for polynomial approximations of degree ≥ 3. For skewed meshes, we can prove that under certain conditions the error can converge optimally. However, in practical computational applications, we expect such cases to be rare and one should generally expect suboptimal convergence rates in $L^2(\Omega)$.

This is not the first DGM that uses second order derivatives. In the works by Engel et al. [11,12], these derivatives are also incorporated by adding Galerkin Least Squares (GLS) stabilization terms to existing DGM's. The DGM introduced in this paper starts with GLS terms on each element that are the variational equivalent of a model reaction-diffusion problem. The final formulation is then derived by applying classical Green's identities (see proof of Lemma 2) to the GLS terms and enforcing continuity across the element interfaces and boundary conditions on the outer boundary.

The outline of this paper is as follows: in Section 2, the model problem, notations, the proper function space setting, and the new DGM are introduced. Subsequently, the convergence properties of the DGM are proved in Section 3. In Section 4, the theoretical results are confirmed on one- and two-dimensional problems. Section 5 concludes with a brief summary of results.

2 The New DG Formulation

2.1 Model Problem and Notations

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain with Lipschitz boundary $\partial\Omega$, and

$$\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}, \quad \Gamma_D \cap \Gamma_N = \varnothing,$$

where Γ_D denotes the part of the boundary with prescribed Dirichlet boundary conditions and Γ_N the part subjected to flux, or Neumann, conditions. Let $\mathcal{P}_h$ be a regular partition of Ω into N open elements $\{K_i\}$ with diameters $\{h_i\}$, such that (see Figure 1):

$$\Omega = \text{int}\left(\bigcup_{i=1}^{N} \overline{K_i}\right). \tag{1}$$

The maximum diameter in the partition is denoted as h and the unit normal vector $\mathbf{n}_i$ on the edge ∂K_i of $K_i \in \mathcal{P}_h$ is directed outward with respect to the element K_i (see Figure 1). Given an element $K_i \in \mathcal{P}_h$, the part of the boundary of ∂K_i that is shared with a neighboring element K_j is denoted ∂K_{ij}, i.e.

$$\partial K_{ij} = \partial K_i \cap \partial K_j. \tag{2}$$

Note that ∂K_{ij} is an *open* subset of ∂K_i. We now consider the following reaction-diffusion-type model problem:

$$
\boxed{
\begin{aligned}
&\text{Find } u, \text{ such that}: \\[1em]
&-\Delta u + u = f, \ \text{ in } \Omega \\[0.5em]
& u = u_0, \ \text{ on } \Gamma_D \\[0.5em]
&\boldsymbol{\nabla} u \cdot \mathbf{n} = g, \ \text{ on } \Gamma_N
\end{aligned}
}
\tag{3}
$$
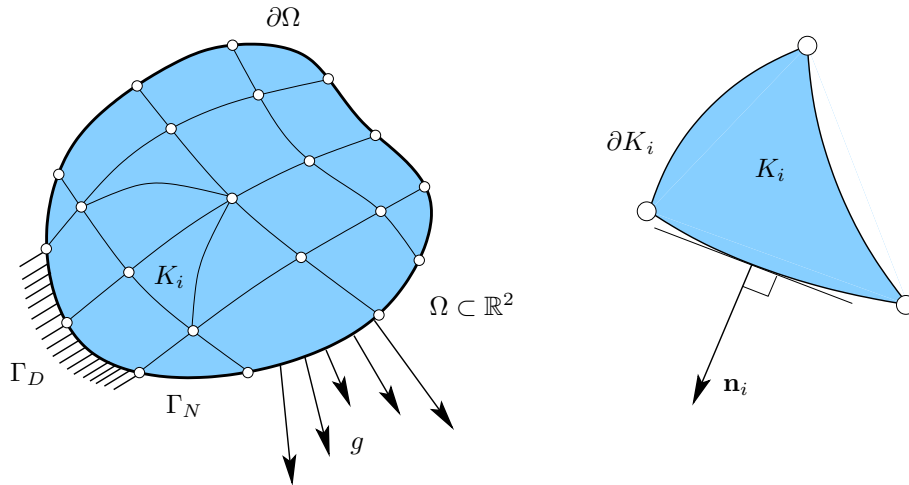

Fig. 1. Geometrical definitions

where f, the source term, is a real-valued function in $L^2(\Omega)$, and the Dirichlet and Neumann boundary data, u_0 and g, are respectively in $H^{3/2}(\partial K_i \cap \Gamma_D)$ and in $H^{1/2}(\partial K_i \cap \Gamma_N)$, for all $K_i \in \mathcal{P}_h$ whose edges intersect with $\partial \Omega$.

2.2 The Broken Banach Space of Test Functions

Given the partition $\mathcal{P}_h$, the following *broken space* is defined:

$$H^2(\mathcal{P}_h) = \left\{ v \in L^2(\Omega) : v_{|K_i} \in H^2(K_i), \ \forall K_i \in \mathcal{P}_h \right\}, \qquad (4)$$

We introduce two norms on this space,

$$
\begin{aligned}
\|v\|^2_{H^2(\mathcal{P}_h)} &= \sum_{i=1}^{N} \left\{ \|\boldsymbol{\nabla}^2 v\|^2_{L^2(K_i)} + 2 \|\boldsymbol{\nabla} v\|^2_{L^2(K_i)} + \|v\|^2_{L^2(K_i)} \right\}, \\
\|v\|^2 &= \sum_{i=1}^{N} \left\{ \|\Delta v\|^2_{L^2(K_i)} + 2 \|\boldsymbol{\nabla} v\|^2_{L^2(K_i)} + \|v\|^2_{L^2(K_i)} \right\},
\end{aligned}
\qquad (5)
$$

where the first uses the local Sobolev norms in $H^2(K_i)$ and the latter the Laplacian norms. Within this setting, we introduce the local zeroth and first order trace operators [13,14] for functions on $K_i \in \mathcal{P}_h$:

$$
\begin{aligned}
\gamma_0^i &: \ H^1(K_i) \longrightarrow H^{1/2}(\partial K_i), \\
\gamma_1^i &: \ H^2(K_i) \longrightarrow H^{1/2}(\partial K_i),
\end{aligned}
\qquad (6)
$$

where $\gamma_1^i(v_i)$ represents the trace of the normal derivative $\partial v_i / \partial n$ on ∂K_i. We define the norm on $H^{1/2}(\partial K_i)$ as follows [13]:

$$\|\varphi\|_{H^{1/2}(\partial K_i)} \overset{\text{def}}{=} \inf_{\substack{v \in H^1(K_i) \\ \gamma_0^i(v) = \varphi}} \|v\|_{H^1(K_i)},$$

Remark 1 *(Trace Inequalities) Let $\mathcal{P}_h$ consist of elements $\{K_i\}$ with Lipschitz boundaries. Then, the trace operators (6) are continuous and surjective (see [13]),*

$$
\begin{aligned}
\|\gamma_0^i(v_i)\|_{H^{1/2}(\partial K_i)} &\leq \|v_i\|_{H^1(K_i)}, \\
\|\gamma_1^i(v_i)\|_{H^{1/2}(\partial K_i)} &\leq C \|v_i\|_{H^2(K_i)}, \quad C > 0.
\end{aligned}
\qquad (7)
$$

4

2.3 The Weak Formulation

The discontinuous variational formulation is stated as follows:

$$
\boxed{
\begin{aligned}
&\text{Find } w \in H^2(\mathcal{P}_h): \\
&\qquad\qquad B(w,v) = L(v), \quad \forall v \in H^2(\mathcal{P}_h),
\end{aligned}
}
\tag{8}
$$

where the bilinear form $B(\cdot,\cdot)$ and linear form $L(\cdot)$ are defined as:

$$
B: H^2(\mathcal{P}_h) \times H^2(\mathcal{P}_h) \longrightarrow \mathbb{R}, \qquad L: H^2(\mathcal{P}_h) \longrightarrow \mathbb{R},
$$

$$
\begin{aligned}
B(w,v) = \sum_{i=1}^{N} \Bigg\{ & \int_{K_i} \Big[\Delta w_i \Delta v_i + 2\, \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla} v_i + w_i v_i \Big]\, \mathrm{d}\mathbf{x} \\
& + \sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \Big[\gamma_1^j(w_j)\gamma_0^i(v_i) - \gamma_1^j(v_j)\gamma_0^i(w_i) \Big]\, \mathrm{d}s \\
& - \int_{\partial K_i \cap \Gamma_N} \gamma_1^i(v_i)\, \gamma_0^i(w_i)\, \mathrm{d}s - \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(w_i)\, \gamma_0^i(v_i)\, \mathrm{d}s \Bigg\},
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
L(v) = \sum_{i=1}^{N} \Bigg\{ & \int_{K_i} f(-\Delta v_i + v_i)\, \mathrm{d}\mathbf{x} + \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(v_i)\, u_0\, \mathrm{d}s \\
& + \int_{\partial K_i \cap \Gamma_N} g\, \gamma_0^i(v_i)\, \mathrm{d}s \Bigg\},
\end{aligned}
$$

where v_i denotes $v_{|K_i}$. Analogous to the DG formulation by Oden, Babuška, and Baumann [4], this formulation satisfies local balance of conservation laws. By taking a function v such that $v = 1$ on an element $K_i \in \mathcal{P}_h$ and $v = 0$ elsewhere, (8) and (9) give us:

$$
\int_{K_i} w_i\, \mathrm{d}\mathbf{x} - \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(w_i)\, \mathrm{d}s = \int_{K_i} f\, \mathrm{d}\mathbf{x} - \sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \gamma_1^j(w_j)\, \mathrm{d}s + \int_{\partial K_i \cap \Gamma_N} g\, \mathrm{d}s.
$$

Let w be a solution of (8). Then the left hand side of the above expression represents the reaction 'forces' of the element to external body and boundary loads (right-hand side). Moreover, we can prove that w satisfies the PDE (3) and belongs to the following space:

$$
H(\Delta,\Omega) = \Big\{ v \in L^2(\Omega): \ \Delta v \in L^2(\Omega) \Big\}.
$$

Lemma 1 *Let $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial K_i \cap \Gamma_N)$ and $u_0 \in H^{3/2}(\partial K_i \cap \Gamma_D)$, and suppose $w \in H^2(\mathcal{P}_h)$ is a solution to the discontinuous Variational Boundary*

Value Problem (VBVP) (8). *Then, w satisfies the PDE* (3) *in the distributional sense and belongs to* $H(\Delta, \Omega) \cap H^2(\mathcal{P}_h)$.

Proof: By taking an arbitrary $K_i \in \mathcal{P}_h$ and choosing a smooth test function v that vanishes outside K_i, then (8) and (9) yield:

$$\int_{K_i} \left[\Delta w_i \Delta \varphi_i + 2 \, \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla} \varphi_i + w_i \varphi_i \right] \mathrm{d}\mathbf{x} = \int_{K_i} f(-\Delta \varphi_i + \varphi_i) \, \mathrm{d}\mathbf{x}.$$

Application of Green's first identity to the integral of $2 \, \boldsymbol{\nabla} w_i \cdot \boldsymbol{\nabla} \varphi_i$, leads to:

$$\int_{K_i} (-\Delta w_i + w_i) \, (-\Delta \varphi_i + \varphi_i) \, \mathrm{d}\mathbf{x} = \int_{K_i} f(-\Delta \varphi_i + \varphi_i) \, \mathrm{d}\mathbf{x}, \quad \forall \varphi_i \in \mathcal{D}(K_i).$$

Thus, in the distributional sense, w satisfies the following problem on any $K_i \in \mathcal{P}_h$:

$$\boxed{-\Delta w_i + w_i = f.} \tag{10}$$

We return to (8) and (9) and again consider an arbitrary element $K_i \in \mathcal{P}_h$ and the interface ∂K_{ij} of this element with one of its neighbors K_j. We choose test functions φ that vanish outside of K_i and everywhere on ∂K_i, except on the element interface ∂K_{ij}. Substituting such test functions and then applying Green's first identity, yields:

$$\int_{K_i} (-\Delta w + w) \, (-\Delta \varphi_i + \varphi_i) \, \mathrm{d}\mathbf{x} + \int_{\partial K_{ij}} \left[\gamma_1^i(w_i) + \gamma_1^j(w_j) \right] \gamma_0^i(\varphi_i) \, \mathrm{d}s$$

$$+ \int_{\partial K_{ij}} \gamma_1^i(\varphi_i) \left[\gamma_0^i(w_i) - \gamma_0^j(w_j) \right] \, \mathrm{d}s = \int_{K_i} f(-\Delta \varphi_i + \varphi_i) \, \mathrm{d}\mathbf{x}.$$

By recalling (10), this expression gives the weak continuity of w and its normal flux $\partial w / \partial n$ across the element interface ∂K_{ij}:

$$\boxed{\gamma_0^i(w_i) = \gamma_0^j(w_j), \qquad \gamma_1^i(w_i) = -\gamma_1^j(w_j).}$$

Obviously, by repeating this procedure we establish the weak continuity of w and $\partial w / \partial n$ across any element interface ∂K_{ij} in the partition $\mathcal{P}_h$, which implies that w is in $H(\Delta, \Omega)$. To prove satisfaction of the Neumann boundary condition, we take test functions φ that vanish outside of K_i and everywhere on ∂K_i except on that part of ∂K_i that coincides with the boundary Γ_N. By using such test functions, we now get:

$$\int_{K_i} (-\Delta w + w) \, (-\Delta \varphi_i + \varphi_i) \, \mathrm{d}\mathbf{x} + \int_{\partial K_i \cap \Gamma_N} \gamma_1^i(w_i) \, \gamma_0^i(\varphi_i) \, \mathrm{d}s$$

$$= \int_{K_i} f(-\Delta \varphi_i + \varphi_i) \, \mathrm{d}\mathbf{x}, + \int_{\partial K_i \cap \Gamma_N} g \gamma_0^i(\varphi_i) \, \mathrm{d}s.$$

Again, recalling (10) reveals that the Neumann boundary condition on $\partial K_i \cap \Gamma_N$ is satisfied weakly:

$$\int_{\partial K_i \cap \Gamma_N} \gamma_1^i(w_i)\, \gamma_0^i(\varphi_i)\, \mathrm{ds} = \int_{\partial K_i \cap \Gamma_N} g \gamma_0^i(\varphi_i)\, \mathrm{ds}.$$

Analogously, we can prove satisfaction of the Dirichlet condition. ∎

In the next lemma, we prove the converse of Lemma 1 and, therefore, establish equivalence between the weak and strong formulation of the model problem.

Lemma 2 *Let $u \in H(\Delta, \Omega) \cap H^2(\mathcal{P}_h)$ be the solution to problem (3), then u is a solution to the VBVP (8).*

Proof: By taking an arbitrary test function $v_i \in H^2(K_i)$, multiplying (3) by $(-\Delta v_i + v_i)$, and integrating over K_i, gives us the Galerkin Least Squares (GLS) representation of (3) on K_i:

$$\int_{K_i} (-\Delta u_i + u_i)(-\Delta v_i + v_i)\, \mathrm{dx} = \int_{K_i} f(-\Delta v_i + v_i)\, \mathrm{dx}.$$

Summing the contributions for all elements in $\mathcal{P}_h$, yields:

$$\sum_{i=1}^N \int_{K_i} (-\Delta u_i + u_i)(-\Delta v_i + v_i)\, \mathrm{dx} = \sum_{i=1}^N \int_{K_i} f(-\Delta v_i + v_i)\, \mathrm{dx}.$$

By applying Green's first identity to the integrals of $u_i \Delta v_i$ and $\Delta u_i v_i$, we get:

$$\sum_{i=1}^N \left\{ \int_{K_i} [\Delta u_i \Delta v_i + 2\boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v_i + u_i v_i]\, \mathrm{dx} \right.$$
$$\left. - \int_{\partial K_i} \left[\gamma_1^i(u_i)\gamma_0^i(v_i) + \gamma_1^i(v_i)\gamma_0^i(u_i) \right]\, \mathrm{ds} \right\} = \sum_{i=1}^N \int_{K_i} f(-\Delta v_i + v_i)\, \mathrm{dx}.$$
$$(11)$$

Concentrating on the edge integrals and applying the boundary conditions,

$$\gamma_0^i(u_i) = u_0, \quad \text{on } \partial K_i \cap \Gamma_D,$$
$$\gamma_1^i(u_i) = g, \quad \text{on } \partial K_i \cap \Gamma_N,$$

leads to:

$$\int_{\partial K_i} \left[\gamma_1^i(u_i)\gamma_0^i(v_i) + \gamma_1^i(v_i)\gamma_0^i(u_i) \right]\, \mathrm{ds} = \int_{\partial K_i \backslash \Gamma_N} \gamma_1^i(u_i)\, \gamma_0^i(v_i)\, \mathrm{ds}$$
$$+ \int_{\partial K_i \backslash \Gamma_D} \gamma_1^i(v_i)\, \gamma_0^i(u_i)\, \mathrm{ds} + \int_{\partial K_i \cap \Gamma_N} g\, \gamma_0^i(v_i)\, \mathrm{ds} + \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(v_i)\, u_0\, \mathrm{ds}.$$
$$(12)$$

Since

$$\int_{\partial K_i \setminus \Gamma_N} \gamma_1^i(u_i)\,\gamma_0^i(v_i)\,\mathrm{ds} = \sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \gamma_1^i(u_i)\,\gamma_0^i(v_i)\,\mathrm{ds}$$
$$+ \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(u_i)\,\gamma_0^i(v_i)\,\mathrm{ds},$$

$$\int_{\partial K_i \setminus \Gamma_D} \gamma_1^i(v_i)\,\gamma_0^i(u_i)\,\mathrm{ds} = \sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \gamma_1^i(v_i)\,\gamma_0^i(u_i)\,\mathrm{ds}$$
$$+ \int_{\partial K_i \cap \Gamma_N} \gamma_1^i(v_i)\,\gamma_0^i(u_i)\,\mathrm{ds},$$

we can rewrite the right-hand-side of (12), which gives:

$$\int_{\partial K_i} \left[\gamma_1^i(u_i)\gamma_0^i(v_i) + \gamma_1^i(v_i)\gamma_0^i(u_i) \right] \mathrm{ds} =$$
$$\sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \left[\gamma_1^i(u_i)\,\gamma_0^i(v_i) + \gamma_1^i(v_i)\,\gamma_0^i(u_i) \right] \mathrm{ds} + \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(u_i)\,\gamma_0^i(v_i)\,\mathrm{ds}$$
$$+ \int_{\partial K_i \cap \Gamma_N} \gamma_1^i(v_i)\,\gamma_0^i(u_i)\,\mathrm{ds} + \int_{\partial K_i \cap \Gamma_N} g\,\gamma_0^i(v_i)\,\mathrm{ds} + \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(v_i)\,u_0\,\mathrm{ds}.$$
$$(13)$$

The solution u to (3) is in $H(\Delta, \Omega)$ and, therefore, its trace and normal derivatives across the element interfaces ∂K_{ij} are continuous, *i.e.*

$$\gamma_0^i(u_i) = \gamma_0^j(u_j), \quad \gamma_1^i(u_i) = -\gamma_1^j(u_j), \quad \text{on every } \partial K_{ij},$$

where u_j denotes the restriction of u to neighboring K_j, and $\gamma_0^j(u_j)$ and $\gamma_1^j(u_j)$ are the traces of u_j on ∂K_j. Implementing these continuity conditions in (13), yields:

$$\int_{\partial K_i} \left[\gamma_1^i(u_i)\gamma_0^i(v_i) + \gamma_1^i(v_i)\gamma_0^i(u_i) \right] \mathrm{ds} =$$
$$- \sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \left[\gamma_1^j(u_j)\,\gamma_0^i(v_i) - \gamma_1^i(v_i)\,\gamma_0^j(u_j) \right] \mathrm{ds} + \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(u_i)\,\gamma_0^i(v_i)\,\mathrm{ds}$$
$$+ \int_{\partial K_i \cap \Gamma_N} \gamma_1^i(v_i)\,\gamma_0^i(u_i)\,\mathrm{ds} + \int_{\partial K_i \cap \Gamma_N} g\,\gamma_0^i(v_i)\,\mathrm{ds} + \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(v_i)\,u_0\,\mathrm{ds}.$$

By substituting this result back into (11) and by noting that

$$\sum_{i=1}^{N} \sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \gamma_1^i(v_i)\,\gamma_0^j(u_j)\,\mathrm{ds} = \sum_{i=1}^{N} \sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \gamma_1^j(v_j)\,\gamma_0^i(u_i)\,\mathrm{ds},$$

we finally get:

$$\sum_{i=1}^{N} \left\{ \int_{K_i} \left[\Delta u_i \Delta v_i + 2 \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v_i + u_i v_i \right] \, \mathrm{d}\mathbf{x} \right.$$

$$+ \sum_{\partial K_{ij} \subset \partial K_i} \int_{\partial K_{ij}} \left[\gamma_1^j(u_j) \, \gamma_0^i(v_i) - \gamma_1^j(v_j) \, \gamma_0^i(u_i) \right] \, \mathrm{d}s$$

$$\left. - \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(u_i) \, \gamma_0^i(v_i) \, \mathrm{d}s - \int_{\partial K_i \cap \Gamma_N} \gamma_1^i(v_i) \, \gamma_0^i(u_i) \, \mathrm{d}s \right\}$$

$$= \sum_{i=1}^{N} \left\{ \int_{K_i} f \left(-\Delta v_i + v_i \right) \, \mathrm{d}\mathbf{x} + \int_{\partial K_i \cap \Gamma_N} g \, \gamma_0^i(v_i) \, \mathrm{d}s + \int_{\partial K_i \cap \Gamma_D} \gamma_1^i(v_i) \, u_0 \, \mathrm{d}s \right\}.$$

which establishes the assertion. ∎

The proof of Lemma 2 uses a GLS representation of (3) on every element K_i. Engel et al. [11,12] have used such GLS terms in their DGM formulations as a stabilization to classical DGM's (e.g. [4,9,15,5]). Here, however, they serve as a starting point in the derivation of a DGM formulation, whose final form is obtained by applying Green's identities to enforce continuity and boundary conditions.

2.4 Continuity and Coercivity Properties

The functionals $B(\cdot, \cdot)$ and $L(\cdot)$ satisfy continuity and coercivity properties on the space $H^2(\mathcal{P}_h)$. We start with an important coercivity property of the bilinear form $B(\cdot, \cdot)$ on $H^2(\mathcal{P}_h) \times H^2(\mathcal{P}_h)$.

Lemma 3 *Let $B(\cdot, \cdot)$ be the bilinear form defined in (9). Then, $B(\cdot, \cdot)$ is coercive with respect to the broken Laplacian norm $\|\cdot\|$,*

$$B(v, v) \geq \frac{1}{2} \|v\|^2, \qquad \forall v \in H^2(\mathcal{P}_h).$$

Proof: Taking $w = v$ in (9), yields:

$$B(v, v) = \sum_{i=1}^{N} \left\{ \|\Delta v_i\|_{L^2(K_i)}^2 + 2 \|\boldsymbol{\nabla} v_i\|_{L^2(K_i)}^2 + \|v_i\|_{L^2(K_i)}^2 \right.$$

$$\left. - \int_{\partial K_i \cap (\Gamma_N \cup \Gamma_D)} \gamma_1^i(v_i) \, \gamma_0^i(v_i) \, \mathrm{d}s \right\}.$$

Since $\gamma_0^i(v_i) \in H^{1/2}(\partial K_i)$ and $\gamma_1^i v_i \in H^{-1/2}(\partial K_i)$ for all $K_i \in \mathcal{P}_h$, the Cauchy-Schwarz Inequality can be applied to the boundary integrals, leading to:

$$B(v,v) \geq \sum_{i=1}^{N} \left\{ \|\Delta v_i\|_{L^2(K_i)}^2 + 2\|\boldsymbol{\nabla} v_i\|_{L^2(K_i)}^2 + \|v_i\|_{L^2(K_i)}^2 \right.$$
$$\left. -\|\gamma_1^i(v_i)\|_{H^{-1/2}(\partial K_i)} \|\gamma_0^i(v_i)\|_{H^{1/2}(\partial K_i)}, \right\}.$$

Applying Young's inequality, gives:

$$B(v,v) \geq \sum_{i=1}^{N} \left\{ \|\Delta v_i\|_{L^2(K_i)}^2 + 2\|\boldsymbol{\nabla} v_i\|_{L^2(K_i)}^2 + \|v_i\|_{L^2(K_i)}^2 \right.$$
$$\left. -\frac{1}{2}\|\gamma_1^i(v_i)\|_{H^{-1/2}(\partial K_i)}^2 - \frac{1}{2}\|\gamma_0^i(v_i)\|_{H^{1/2}(\partial K_i)}^2 \right\}.$$

Recalling the trace inequality $(7)^1$, yields:

$$B(v,v) \geq \sum_{i=1}^{N} \left\{ \|\Delta v_i\|_{L^2(K_i)}^2 + \frac{3}{2}\|\boldsymbol{\nabla} v_i\|_{L^2(K_i)}^2 + \frac{1}{2}\|v_i\|_{L^2(K_i)}^2 \right. \tag{14}$$
$$\left. -\frac{1}{2}\|\gamma_1^i(v_i)\|_{H^{-1/2}(\partial K_i)}^2 \right\}$$

We call upon Theorem 2.5 in [13] and state the following trace inequality for $v \in H^2(K_i)$:

$$\|\gamma_1^i(v_i)\|_{H^{-1/2}(\partial K_i)}^2 \leq \|\Delta v_i\|_{L^2(K_i)}^2 + \|\boldsymbol{\nabla} v_i\|_{L^2(K_i)}^2. \tag{15}$$

Substituting this inequality into (14) establishes the assertion. ∎

We cannot prove coercivity of the bilinear form with respect to the broken Sobolev norm $\|\cdot\|_{H^2(\mathcal{P}_h)}$. With the issue of continuity, we face a converse situation: continuity of $B(\cdot, \cdot)$ on $H^2(\mathcal{P}_h) \times H^2(\mathcal{P}_h)$ can be proved in the norm $\|\cdot\|_{H^2(\mathcal{P}_h)}$ but not in the norm $\|\|\cdot\|\|$.

Lemma 4 *The bilinear form $B(\cdot, \cdot)$ is continuous on $H^2(\mathcal{P}_h) \times H^2(\mathcal{P}_h)$, i.e. there exists a constant $M > 0$ such that:*

$$\boxed{|B(w,v)| \leq M \|w\|_{H^2(\mathcal{P}_h)} \|v\|_{H^2(\mathcal{P}_h)} \qquad \forall w, v \in H^2(\mathcal{P}_h).}$$

Proof: Since $w, v \in H^2(\mathcal{P}_h)$, the zeroth and first order traces of these functions are respectively in $H^{3/2}(\partial K_i)$ and $H^{1/2}(\partial K_i)$, which are both subspaces of

$L^2(\partial K_i)$. Thus, we can apply the Cauchy-Schwarz inequality to $B(\cdot, \cdot)$ in the following manner:

$$B(w, v) \leq \sum_{i=1}^{N} \Bigg\{ \|\Delta w_i\|_{L^2(K_i)} \|\Delta v_i\|_{L^2(K_i)} + 2 \|\boldsymbol{\nabla} w_i\|_{L^2(K_i)} \|\boldsymbol{\nabla} v_i\|_{L^2(K_i)}$$

$$+ \|w_i\|_{L^2(K_i)} \|v_i\|_{L^2(K_i)} + \sum_{\partial K_{ij} \subset \partial K_i} \Big\{ \|\gamma_1^j(w_j)\|_{L^2(\partial K_{ij})} \|\gamma_0^i(v_i)\|_{L^2(\partial K_{ij})}$$

$$+ \|\gamma_1^j(v_j)\|_{L^2(\partial K_{ij})} \|\gamma_0^i(w_i)\|_{L^2(\partial K_{ij})} \Big\} + \|\gamma_1^i(v_i)\|_{L^2(\partial K_i \cap \Gamma_N)} \|\gamma_0^i(w_i)\|_{L^2(\partial K_i \cap \Gamma_N)}$$

$$+ \|\gamma_1^i(w_i)\|_{L^2(\partial K_i \cap \Gamma_D)} \|\gamma_0^i(v_i)\|_{L^2(\partial K_i \cap \Gamma_D)} \Bigg\}.$$

We can bound this inequality as follows:

$$B(w, v) \leq C \sqrt{\|w\|^2 + \sum_{i=1}^{N} \Big\{ \|\gamma_1^i(w_i)\|_{L^2(\partial K_i)}^2 + \|\gamma_0^i(w_i)\|_{L^2(\partial K_i)}^2 \Big\}}$$

$$\times \sqrt{\|v\|^2 + \sum_{i=1}^{N} \Big\{ \|\gamma_1^i(v_i)\|_{L^2(\partial K_i)}^2 + \|\gamma_0^i(v_i)\|_{L^2(\partial K_i)}^2 \Big\}}, \quad C > 0,$$

where $\|\cdot\|$ is defined as in (5). Since $H^{1/2}(\partial K_i)$ is embedded in $L^2(\partial K_i)$, we can assert that:

$$B(w, v) \leq C \sqrt{\|w\|^2 + \sum_{i=1}^{N} \Big\{ \|\gamma_1^i(w_i)\|_{H^{1/2}(\partial K_i)}^2 + \|\gamma_0^i(w_i)\|_{H^{1/2}(\partial K_i)}^2 \Big\}}$$

$$\times \sqrt{\|v\|^2 + \sum_{i=1}^{N} \Big\{ \|\gamma_1^i(v_i)\|_{H^{1/2}(\partial K_i)}^2 + \|\gamma_0^i(v_i)\|_{H^{1/2}(\partial K_i)}^2 \Big\}}, \quad C > 0.$$

Application of the trace inequalities (7) completes the proof. ∎

Proposition 1 *The linear form $L(\cdot)$ is continuous on $H^2(\mathcal{P}_h)$:*

$$\exists C > 0 : \ L(v) \leq C \|v\|_{H^2(\mathcal{P}_h)}, \qquad \forall v \in H^2(\mathcal{P}_h), \tag{16}$$

where $C = C(f, u_0, g)$.

Remark 2 *(Well Posedness) Although the bilinear form $B(\cdot, \cdot)$ satisfies coercivity and continuity properties, we cannot invoke the Generalized Lax-Milgram Theorem to prove existence of unique solutions of (8), for two-dimensional problems. The coercivity property in Lemma 3 is satisfied in terms of the norm $\|\cdot\|$. For two-dimensional problems, the space $H^2(\mathcal{P}_h)$ is not complete with respect to this norm and completeness is an essential condition in the Lax Milgram Theorem.*

However, for classes of problems of (3) *for which we can prove there exists a solution* $u \in H^2(\mathcal{P}_h) \cap H(\Delta, \Omega)$ *(e.g. via the conventional continuous variational formulation), we know from Lemma 2 that* u *is a solution to the DGM formulation* (8). *Uniqueness is then guaranteed, as the bilinear form is positive definite on* $H^2(\mathcal{P}_h)$.

3 Convergence

3.1 The Discrete Problem

Let $\{\mathbf{F}_{K_i}\}$ be a family of invertible maps defined on the partition $\mathcal{P}_h$ such that every element $K_i \in \mathcal{P}_h$ is the image of $\mathbf{F}_{K_i}$ acting on a master element $\hat{K}$, as shown in Figure 2.

$$\mathbf{F}_{K_i} : \hat{K} \longrightarrow K_i, \qquad\qquad \mathbf{x} = \mathbf{F}_{K_i}(\hat{\mathbf{x}}). \qquad\qquad (17)$$

Unless stated otherwise, the sets of mappings are assumed to be affine. We introduce a finite dimensional space of real-valued piecewise polynomial functions,

$$\mathcal{V}^{hp} = \left\{ v \in L^2(\Omega) : \ v_{|K_i} = \hat{v} \circ \mathbf{F}_{K_i}^{-1} , \ \hat{v} \in P^{p_i}(\hat{K}), \ \forall K_i \in \mathcal{P}_h \right\} \subset H^2(\mathcal{P}_h), \tag{18}$$

where $P^{p_i}(\hat{K})$ denotes the space of polynomials on $\hat{K}$ of degree $\leq p_i$, in which p_i can have different values on different elements. Let $u \in H^2(\mathcal{P}_h)$ be the solution of (8). Then, we seek a discrete approximation $u_h \in \mathcal{V}^{hp}$ by solving
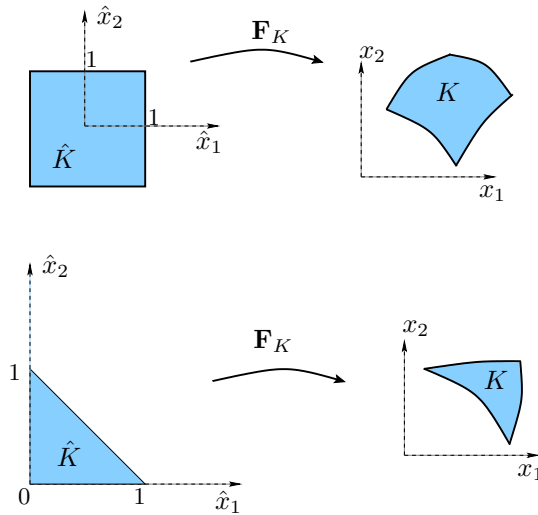

Fig. 2. Mapping from the master elements to the physical space

the following (discrete) variational problem:

$$
\boxed{
\begin{aligned}
&\text{Find } u_h \in \mathcal{V}^{hp} : \\
&\qquad\qquad B(u_h, v_h) = L(v_h), \quad \forall v_h \in \mathcal{V}^{hp}.
\end{aligned}
}
\tag{19}
$$

Lemma 5 *The bilinear form $B(\cdot, \cdot)$ is coercive on $\mathcal{V}^{hp} \times \mathcal{V}^{hp}$ with respect to the norm $\|\cdot\|_{H^2(\mathcal{P}_h)}$ (see (5)), i.e.*

$$
\boxed{
\exists C > 0 : \qquad |B(v_h, v_h)| \geq C \|v_h\|^2_{H^2(\mathcal{P}_h)}, \qquad \forall v_h \in \mathcal{V}^{hp}.
}
$$

Proof: Since $\mathcal{V}^{hp} \subset H^2(\mathcal{P}_h)$, we know from Lemma 3 that the bilinear form $B(\cdot, \cdot)$ is coercive on $\mathcal{V}^{hp} \times \mathcal{V}^{hp}$ with respect to the norm $\|\!|\cdot|\!\|$. For finite dimensional spaces, this norm is equivalent to $\|\cdot\|_{H^2(\mathcal{P}_h)}$. Thus,

$$
\exists C > 0 : \qquad \|v_h\|_{H^2(\mathcal{P}_h)} \leq C \|\!|v_h|\!\|, \qquad \forall v_h \in \mathcal{V}^{hp},
$$

which establishes the assertion. ∎

Since the bilinear form $B(\cdot, \cdot)$ is continuous, coercive, and positive definite on $\mathcal{V}^{hp} \times \mathcal{V}^{hp}$, existence of unique solutions $u_h \in \mathcal{V}^{hp}$ to (19) is established by applying the *Generalized Lax Milgram Theorem*. If u is then the solution to (8), it easily follows that the approximation error $e_h = u - u_h$ is governed by the following variational problem:

$$
\boxed{
\begin{aligned}
&\text{Find } e_h \in H^2(\mathcal{P}_h) \text{ such that} \\
&\qquad B(e_h, v) = \underbrace{L(v) - B(u_h, v)}_{\mathcal{R}_h(v)}, \quad \forall v \in H^2(\mathcal{P}_h)
\end{aligned}
}
\tag{20}
$$

where $\mathcal{R}_h : H^2(\mathcal{P}_h) \longrightarrow \mathbb{R}$ is the *Residual Functional*, which satisfies the Galerkin orthogonality property on the space $\mathcal{V}^{hp}$, *i.e.*

$$
\boxed{
\mathcal{B}(e_h, v_h) = \mathcal{R}_h(v) = 0, \quad \forall v_h \in \mathcal{V}^{hp}.
}
\tag{21}
$$

3.2 A Priori Error Estimates in $H^2(\mathcal{P}_h)$

In this section, we derive convergence rates of the approximation error $e_h = u - u_h$ in terms of the norm $\|\cdot\|_{H^2(\mathcal{P}_h)}$. The convergence rates in lower norms

are derived in the next section. We start by defining a set of interpolants $\{\pi_{hp}^i\}$ for every $\mathcal{P}_h$, such that:

$$\pi_{hp}^i: \ H^{r_i}(K_i) \longrightarrow P^{p_i}(K_i), \quad K_i \in \mathcal{P}_h, \quad i = 1, 2, \ldots, N,$$

$$\pi_{hp}^i(v_h) = v_h, \qquad \forall\, v_h \in P^{p_i}(K_i),$$

where $r_i \geq 2$. We can now call upon an interpolation theorem proved in [16].

Theorem 2 *For $\varphi \in H^{r_i}(K_i)$, there exists $C > 0$, independent of φ, p_i and r_i, and a sequence $\pi_{hp}^i(\varphi) \in P^{p_i}(K_i)$, such that:*

$$
\begin{aligned}
\|\varphi - \pi_{hp}^i(\varphi)\|_{L^2(K_i)} &\leq C \frac{h_i^{\mu_i}}{p_i^{r_i}} \|\varphi\|_{H^{r_i}(K_i)}, \\[2mm]
\|\boldsymbol{\nabla}\varphi - \boldsymbol{\nabla}\pi_{hp}^i(\varphi)\|_{L^2(K_i)} &\leq C \frac{h_i^{\mu_i-1}}{p_i^{r_i-1}} \|\varphi\|_{H^{r_i}(K_i)}, \quad r_i \geq 1,\ p_i \geq 1, \\[2mm]
\|\boldsymbol{\nabla}^2\varphi - \boldsymbol{\nabla}^2\pi_{hp}^i(\varphi)\|_{L^2(K_i)} &\leq C \frac{h_i^{\mu_i-2}}{p_i^{r_i-2}} \|\varphi\|_{H^{r_i}(K_i)}.
\end{aligned}
$$

where $\mu_i = \min(p_i + 1, r_i)$.

By extending the local interpolants $\pi_{hp}^i(.)$ to zero outside of K_i, we can define a global interpolant on the whole partition $\mathcal{P}_h$:

$$\Pi_{hp}: \ H^2(\mathcal{P}_h) \longrightarrow \mathcal{V}^{hp}, \qquad \Pi_{hp}(v) \overset{\text{def}}{=} \sum_{K_i \in \mathcal{P}_h} \pi_{hp}^i(v_{|K_i}), \qquad v \in H^2(\mathcal{P}_h). \quad (22)$$

Lemma 6 *(Interpolation Lemma) Let $u \in H^2(\mathcal{P}_h)$. Then, there exists $C > 0$, independent of u, $\{h_i\}$ and $\{p_i\}$ such that the interpolation error $\eta = u - \Pi_{hp}u$ can be bounded as follows:*

$$\|\eta\|_{H^2(\mathcal{P}_h)} \leq C \frac{h^{\mu-2}}{p^{r-2}} \sqrt{\sum_{K_i \in \mathcal{P}_h} \|u\|_{H^{r_i}(K_i)}^2}, \quad r_i \geq 2,$$

where $r = \min_{K_i \in \mathcal{P}_h} \{r_i\}$, $p = \max_{K_i \in \mathcal{P}_h} \{p_i\}$, $h = \max_{K_i \in \mathcal{P}_h} \{h_i\}$, and $\mu = \min(p+1, r)$.

Proof: The proof of this lemma is quickly established by recalling the definition of the norm $\|\cdot\|_{H^2(\mathcal{P}_h)}$, as given in (5), and substituting the inequalities listed in Theorem 2. ∎

Having established convergence rates for the interpolation error, we can now derive *optimal* convergence rates of the approximation error e_h in the broken space $H^2(\mathcal{P}_h)$.

Theorem 3 *(Convergence) Let $u \in H^2(\mathcal{P}_h)$ be the unique solution to the variational problem* (8) *and* $\{u_h \in \mathcal{V}^{hp}\}$ *be a sequence of approximations* (19) *of* u. *Then, the approximation error* $e_h = u - u_h$ *is bounded as follows:*

$$\|e_h\|_{H^2(\mathcal{P}_h)} \leq C \, \frac{h^{\mu-2}}{p^{r-2}} \sqrt{\sum_{K_i \in \mathcal{P}_h} \|u\|_{H^{r_i}(K_i)}^2}, \quad p_i \geq 1 \; ; \; r_i \geq 2,$$

where $r = \min\limits_{K_i \in \mathcal{P}_h} \{r_i\}$, $p = \max\limits_{K_i \in \mathcal{P}_h} \{p_i\}$, $h = \max\limits_{K_i \in \mathcal{P}_h} \{h_i\}$, *and* $\mu = \min(p + 1, r)$.

Proof: Let $\Pi_{hp}(\cdot)$ be the interpolant operator as defined in (22). Then we introduce two functions η and ξ, such that the approximation error can be written as $e_h = \eta - \xi$, where $\eta = u - \Pi_{hp} u$ and $\xi = u_h - \Pi_{hp} u$. Note that $\xi \in \mathcal{V}^{hp}$ and that the interpolation error η is in $H^2(\mathcal{P}_h)$. By using the triangle inequality, we obtain:

$$\|e_h\|_{H^2(\mathcal{P}_h)} \leq \|\eta\|_{H^2(\mathcal{P}_h)} + \|\xi\|_{H^2(\mathcal{P}_h)}. \tag{23}$$

Recalling the coercivity property of Lemma 5 leads to:

$$\|\xi\|_{H^2(\mathcal{P}_h)}^2 \leq C \, B(\xi, \xi).$$

By applying the orthogonality property (21) and the continuity of the bilinear form $B(\cdot, \cdot)$ (see Lemma 4), we get:

$$\|\xi\|_{H^2(\mathcal{P}_h)}^2 \leq C \, B(\eta, \xi) \leq C \, \|\eta\|_{H^2(\mathcal{P}_h)} \|\xi\|_{H^2(\mathcal{P}_h)}$$

Thus, returning to (23), we can conclude:

$$\|e_h\|_{H^2(\mathcal{P}_h)} \leq C \|\eta\|_{H^2(\mathcal{P}_h)}.$$

We finish the proof by applying Lemma 6. ∎

Remark 3 *Take $p = 1$. From Theorem 3, $\mu = 2$ and $\mu - 2 = 0$, which implies that the approximate solutions $\{u_h\}$ do not converge for h refinements.*

3.3 The Aubin-Nitsche Lift - A Priori Error Estimates in Lower Norms

For $p \geq 2$, convergence to the solution of the target problem (8) is guaranteed by Theorem 3, but the rates are suboptimal in terms of the $H^1(\mathcal{P}_h)$ and $L^2(\Omega)$ norms. We employ a technique introduced by Aubin [17] and Nitsche [18] to prove that, under certain conditions, the approximation error also converges optimally in these lower norms. First, we introduce broken Hilbert spaces on the polygon partitions $\{\mathcal{P}_h\}$:

$$H^\sigma(\mathcal{P}_h) = \left\{ v \in L^2(\Omega) : \; v_{|K_i} \in H^\sigma(K_i), \; \forall K_i \in \mathcal{P}_h \right\}, \quad 0 \leq \sigma \leq 1, \tag{24}$$

on which we define broken Sobolev norms,

$$\|v\|_{H^\sigma(\mathcal{P}_h)} = \sqrt{\sum_{K_i \in \mathcal{P}_h} \|v_i\|_{H^\sigma(K_i)}^2}, \qquad v \in H^\sigma(\mathcal{P}_h). \tag{25}$$

We follow [17,18] by introducing the functionals $q(\cdot)$ in the dual space $H^{-\sigma}(\mathcal{P}_h)$, for which we can prove that there exist Riesz-type representative functions w_q in the following subspace of $H^2(\mathcal{P}_h)$:

$$H_{00}^2(\mathcal{P}_h) = \left\{ v \in H^2(\mathcal{P}_h) : \ \gamma_0^i(v_i) = 0, \ \gamma_1^i(v_i) = 0, \ \forall K_i \in \mathcal{P}_h \right\} \subset H^2(\mathcal{P}_h).$$

Lemma 7 *For every $q \in H^{-\sigma}(\mathcal{P}_h)$, there exists a unique $w_q \in H_{00}^2(\mathcal{P}_h)$, such that:*

$$\boxed{B(v, w_q) = q(v), \qquad \forall v \in H^2(\mathcal{P}_h)} \tag{26}$$

Proof: If we recall the coercivity property of Lemma 3, we get:

$$\sup_{v \in H^2(\mathcal{P}_h)/\{0\}} \frac{|B(v,w)|}{\|v\|_{H^2(\mathcal{P}_h)}} \geq \frac{1}{2} \frac{\|\|w\|\|^2}{\|w\|_{H^2(\mathcal{P}_h)}} \qquad \forall w \in H_{00}^2(\mathcal{P}_h)/\{0\}.$$

For functions that belong to $H_{00}^2(K_i)$, the norm $\|\Delta w_i\|_{L^2(K_i)}$ is equal to the norm $\|\boldsymbol{\nabla}^2 w_i\|_{L^2(K_i)}$ (*e.g.* see [13]). From (5) then follows that the norms $\|\cdot\|_{H^2(\mathcal{P}_h)}$ and $\|\|\cdot\|\|$ are identical for functions that belong to $H_{00}^2(\mathcal{P}_h)$. Hence, the above expression gives us the Inf-Sup condition for the bilinear form $B(\cdot,\cdot)$ on $H^2(\mathcal{P}_h) \times H_{00}^2(\mathcal{P}_h)$:

$$\sup_{v \in H^2(\mathcal{P}_h)/\{0\}} \frac{|B(v,w)|}{\|v\|_{H^2(\mathcal{P}_h)}} \geq \frac{1}{2} \|w\|_{H^2(\mathcal{P}_h)}, \qquad \forall w \in H_{00}^2(\mathcal{P}_h)/\{0\}.$$

Considering that the bilinear form is also positive definite and continuous (see Lemma 4), we can call upon the *Generalized Lax Milgram Theorem* to assert that there exists a unique solution $w_q \in H_{00}^2(\mathcal{P}_h)$. ∎

By duality, the norm of the error in the spaces $H^\sigma(\mathcal{P}_h)$, $0 \leq \sigma \leq 1$, is closely related to the functions w_q, *i.e.*

$$\|e_h\|_{H^\sigma(\mathcal{P}_h)} \stackrel{\text{def}}{=} \sup_{q \in H^{-\sigma}(\mathcal{P}_h)} \frac{|q(e_h)|}{\|q\|_{H^{-\sigma}(\mathcal{P}_h)}} = \sup_{q \in H^{-\sigma}(\mathcal{P}_h)} \frac{|B(e_h, w_q)|}{\|q\|_{H^{-\sigma}(\mathcal{P}_h)}}.$$

Let $\Pi_{hp}(\cdot) \in \mathcal{V}^{hp}$ denote the global interpolation operator as defined in (22). Then, by applying the Galerkin orthogonality property (21) and continuity of

16

the bilinear form, we can rewrite this expression as:

$$\|e_h\|_{H^\sigma(\mathcal{P}_h)} \leq C \, \|e_h\|_{H^2(\mathcal{P}_h)} \sup_{q \in H^{-\sigma}(\mathcal{P}_h)} \frac{\|w_q - \Pi_{hp} w_q\|_{H^2(\mathcal{P}_h)}}{\|q\|_{H^{-\sigma}(\mathcal{P}_h)}}, \quad C > 0.$$

By applying the interpolation Lemma 6 and convergence Theorem 3, we can further bound the error,

$$\|e_h\|_{H^\sigma(\mathcal{P}_h)} = C \frac{h^{\mu+\nu-4}}{p^{r+s-4}} \sqrt{\sum_{K_i \in \mathcal{P}_h} \|u\|_{H^{r_i}(K_i)}^2} \sup_{q \in H^{-\sigma}(\mathcal{P}_h)} \frac{\sqrt{\sum_{K_i \in \mathcal{P}_h} \|w_q\|_{H^{s_i}(K_i)}^2}}{\|q\|_{H^{-\sigma}(\mathcal{P}_h)}}, \quad (27)$$

where h, p, r, r_i, and μ are defined in Theorem 6, and $s = \min_{K_i \in \mathcal{P}_h} \{s_i\}$ and $\nu = \min(p+1, s)$. So convergence of the approximation error in the lower $H^1(\mathcal{P}_h)$ and $L^2(\Omega)$ norms is governed by the regularity of the solutions w_q. To determine this regularity, we call upon a regularity theorem that is based on the work on polygonal domains by Grisvard [19].

Lemma 8 *Let each partition $\mathcal{P}_h$ consist of convex polygons $\{K_i\}$, $i = 1, 2, \ldots, N$, and each element K_i have N_c^i corners with angles ω_j^i, $j = 1, 2, \ldots, N_c^i$ (see Figure 3). Let w_q be the solution to (26) for a given $q \in H^{-\sigma}(\mathcal{P}_h)$, $0 \leq \sigma \leq 1$. If the following characteristic equations each have no root $\lambda \in \mathbb{C}$ other than $\lambda = -I$ on the line $(\sigma - 2)I$ (where I denotes the imaginary variable):*

$$\sinh^2(\lambda \omega_j^i) = \lambda^2 \sin^2(\omega_j^i), \qquad j = 1, \ldots, N_c^i, \quad i = 1, \ldots, N. \qquad (28)$$

Then, every w_q belongs to $H^{4-\sigma}(\mathcal{P}_h)$. Otherwise, the functions w_q are in $H^{3-\sigma}(\mathcal{P}_h)$.

Proof: This lemma is a result of Theorem 7.2.2.3 and Remark 7.2.2.4 in the work by Grisvard [19]. These establish the regularity of solutions to the biharmonic equation and can be applied to establish the regularity of the func-
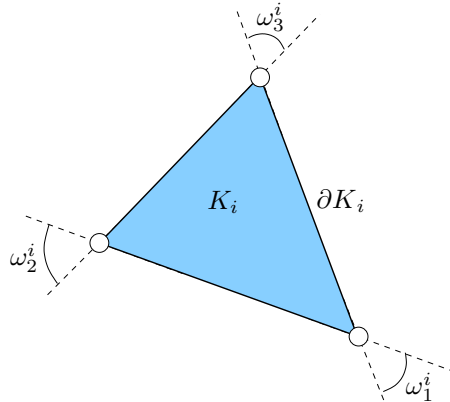

Fig. 3. Corner angles of the polygons.

tions w_q, as these satisfy a bi-harmonic equation on each element K_i:

$$\Delta\Delta w_q = \Phi, \qquad \forall K_i \in \mathcal{P}_h, \tag{29}$$

where $\Phi \in H^{-\sigma}(K_i)$ and homogeneous Dirichlet and Neumann conditions hold on ∂K_i. To prove this assertion, we start by substituting test functions $v = \varphi$ into (26) whose restrictions φ_i to K_i belong to $\mathcal{D}(K_i)$. Thus, we get

$$\langle \Delta\varphi_i,\, \Delta w_q \rangle + 2\langle \boldsymbol{\nabla}\varphi_i,\, \boldsymbol{\nabla}w_q \rangle + \langle \varphi_i,\, w_q \rangle = \langle \varphi_i,\, q \rangle \varphi_i, \qquad \forall \varphi_i,\, \forall K_i \in \mathcal{P}_h,$$

where $\langle \cdot,\, \cdot \rangle$ denotes the duality pairing in $\mathcal{D}(K_i) \times \mathcal{D}(K_i)'$. Application of the definition of the distributional derivative, then gives:

$$\langle \varphi_i,\, \Delta\Delta w_q - 2\Delta w_q + w_q \rangle = \langle \varphi_i,\, q \rangle, \qquad \forall \varphi_i,\, \forall K_i \in \mathcal{P}_h.$$

Comparison with (29) reveals that $\Phi = q + 2\Delta w_q - w_q$. Since $q \in H^{-\sigma}(K_i)$ and $\Delta w_q, w_q \in L^2(K_i)$, Φ must belong to $H^{-\sigma}(K_i)$. The proof of this lemma is completed by applying Theorem 7.2.2.3 and Remark 7.2.2.4 in [19] to (29) and by noting that the functions w_q satisfy homogeneous Dirichlet and Neumann boundary conditions on ∂K_i and that every polygon K_i is convex (*i.e.* no re-entrant corners). ∎

We first use the result of Lemma 8 to derive h and p convergence rates in the broken space $H^1(\mathcal{P}_h)$.

Theorem 4 (*Convergence in $H^1(\mathcal{P}_h)$*) *Let $u \in H^2(\mathcal{P}_h)$ be the unique solution to the variational problem* (8) *and $\{u_h \in \mathcal{V}^{hp}\}$ be a sequence of approximations* (19) *using families of regular partitions $\{\mathcal{P}_h\}$ of convex polygons. Then, the approximation error $e_h = u - u_h$ is bounded in the $H^1(\mathcal{P}_h)$ norm* (25)*, i.e.*

$$\exists C > 0: \qquad \|e_h\|_{H^1(\mathcal{P}_h)} \leq C\, \frac{h^{\mu-1}}{p^{r-1}} \sqrt{\sum_{K_i \in \mathcal{P}_h} \|u\|^2_{H^{r_i}(K_i)}}, \qquad p \geq 2,\ r_i \geq 2,$$

where $r = \min\limits_{K_i \in \mathcal{P}_h} \{r_i\}$, $p = \max\limits_{K_i \in \mathcal{P}_h} \{p_i\}$, $h = \max\limits_{K_i \in \mathcal{P}_h} \{h_i\}$, and $\mu = \min(p+1, r)$.

Proof: We return to (27) and set $\sigma = 1$:

$$\|e_h\|_{H^1(\mathcal{P}_h)} = C\, \frac{h^{\mu+\nu-4}}{p^{r+s-4}} \sqrt{\sum_{K_i \in \mathcal{P}_h} \|u\|^2_{H^{r_i}(K_i)}} \ \sup_{q \in H^{-1}(\mathcal{P}_h)} \frac{\sqrt{\sum_{K_i \in \mathcal{P}_h} \|w_q\|^2_{H^{s_i}(K_i)}}}{\|q\|_{H^{-1}(\mathcal{P}_h)}}. \tag{30}$$

We now call upon Lemma 8 to determine the regularity of the functions w_q, which means that we have to find the roots $\lambda \in \mathbb{C}$ to the following character-

istic equations on the line $-I$:

$$\sinh(\lambda \omega_j^i) = \pm \lambda \sin(\omega_j^i), \qquad j = 1, \ldots, N_c^i, \quad i = 1, \ldots, N.$$

Lemma 7.3.2.4 in [19] reveals that the above equations have only one root $\lambda = -I$ on the line $-I$. From Lemma 8 we then conclude that the functions w_q are in $H^3(\mathcal{P}_h)$. This implies that $s \geq 3$ and, with $p \geq 2$, we obtain $\nu \geq 3$. Returning to (30), this gives:

$$\|e_h\|_{H^1(\mathcal{P}_h)} \leq C \frac{h^{\mu-1}}{p^{r-1}} \sqrt{\sum_{K_i \in \mathcal{P}_h} \|u\|^2_{H^{r_i}(K_i)}} \sup_{q \in H^{-1}(\mathcal{P}_h)} \frac{\|w_q\|_{H^3(\mathcal{P}_h)}}{\|q\|_{H^{-1}(\mathcal{P}_h)}}, \qquad (31)$$

To bound the term involving the supremum, we return to (26) and rewrite this as follows:

$$\langle v, \mathbb{B}^* w_q \rangle = \langle v, q \rangle, \qquad \forall v \in H^2(\mathcal{P}_h),$$

where $\mathbb{B}^* : H_{00}^2(\mathcal{P}_h) \longrightarrow H^{-2}(\mathcal{P}_h)$ is a linear operator associated with the bilinear form $B(\cdot, \cdot)$. Since we know that the bilinear form is continuous on $H^2(\mathcal{P}_h) \times H_{00}^2(\mathcal{P}_h)$, the operator $\mathbb{B}^*(\cdot)$ is continuous on $H_{00}^2(\mathcal{P}_h) \cap H^3(\mathcal{P}_h)$, *i.e.*

$$\|\mathbb{B}^* w_q\|_{H^{-2}(\mathcal{P}_h)} \leq C \|w_q\|_{H^2(\mathcal{P}_h)} \leq C \|w_q\|_{H^3(\mathcal{P}_h)}.$$

We have established that for every $q \in H^{-1}(\mathcal{P}_h)$ there exists a unique $w_q \in H^3(\mathcal{P}_h) \cap H_{00}^2(\mathcal{P}_h)$. Hence, $\mathbb{B}^*(\cdot)$ is bijective from $H^3(\mathcal{P}_h) \cap H_{00}^2(\mathcal{P}_h)$ to $H^{-1}(\mathcal{P}_h) \cap H^{-2}(\mathcal{P}_h)$. Since the restriction of $\mathbb{B}^*(\cdot)$ to $H^3(\mathcal{P}_h)$ is both continuous and bijective, the *Banach Theorem* states that the inverse of the restriction of $\mathbb{B}^*(\cdot)$ is continuous. Thus, there exists $C > 0$, independent of $q(\cdot)$, such that:

$$\|w_q\|_{H^3(\mathcal{P}_h)} = \|\mathbb{B}^{*-1} q\|_{H^3(\mathcal{P}_h)} \leq C \|q\|_{H^{-2}(\mathcal{P}_h)}, \quad \forall w_q \in H^3(\mathcal{P}_h) \cap H_{00}^2(\mathcal{P}_h).$$

By backsubstituting this result into (31), we conclude the proof. ■

Thus, we have proved optimal h and p convergence of the error in $H^1(\mathcal{P}_h)$, for $p \geq 2$, if convex polygonal elements are used. Optimal h and p convergence in $L^2(\Omega)$ is also possible but it depends on the shape of the polygons in the partition $\mathcal{P}_h$, in particular the value of the corner angles. The following theorem is a consequence of Lemma 8 and states a necessary condition on the corners of the polygons in order to obtain optimal convergence rates in $L^2(\Omega)$.

Theorem 5 *(Convergence in $L^2(\Omega)$) Let $u \in H^2(\mathcal{P}_h)$ be the unique solution to the variational problem (8) and $\{u_h \in \mathcal{V}^{hp}\}$ be a sequence of approximations (19) using families of regular partitions $\{\mathcal{P}_h\}$ of convex polygons. Let N_c^i denote the number of corners of an element K_i and ω_j^i denote the jth corner angle of this element. If the following characteristic equations have no roots a*

in $\mathbb{R} \setminus \{0\}$,

$$\tanh(a\,\omega_j^i) = \frac{1}{2}\tan(2\omega_j^i)\,a, \qquad j = 1, 2, \ldots, N_c^i, \quad i = 1, 2, \ldots, N. \qquad (32)$$

Then, the approximation error $e_h = u - u_h$ *is bounded in the* $L^2(\Omega)$ *norm such that* $\exists C > 0$:

$$\|e_h\|_{L^2(\Omega)} \leq C\,\frac{h^{\mu-1}}{p^r}\sqrt{\sum_{K_i \in \mathcal{P}_h}\|u\|^2_{H^{r_i}(K_i)}}, \qquad p = 2,\ r_i \geq 2,$$

$$\|e_h\|_{L^2(\Omega)} \leq C\,\frac{h^\mu}{p^r}\sqrt{\sum_{K_i \in \mathcal{P}_h}\|u\|^2_{H^{r_i}(K_i)}}, \qquad p \geq 3,\ r_i \geq 2,$$

where $r = \min\limits_{K_i \in \mathcal{P}_h}\{r_i\}$, $p = \max\limits_{K_i \in \mathcal{P}_h}\{p_i\}$, $h = \max\limits_{K_i \in \mathcal{P}_h}\{h_i\}$, *and* $\mu = \min(p+1, r)$.
Otherwise, the error is bounded by the following suboptimal bounds:

$$\|e_h\|_{L^2(\Omega)} \leq C\,\frac{h^{\mu-1}}{p^{r-1}}\sqrt{\sum_{K_i \in \mathcal{P}_h}\|u\|^2_{H^{r_i}(K_i)}}, \qquad p \geq 2,\ r_i \geq 2,$$

Proof: Since $\|\cdot\|_{L^2(\Omega)} \leq \|\cdot\|_{H^1(\mathcal{P}_h)}$, the error in $L^2(\Omega)$ converges at least at the same rate as in $H^1(\mathcal{P}_h)$. To obtain a lift in convergence rates, we follow the proof of Theorem 4, recall (27), and set $\sigma = 0$, *i.e.*:

$$\|e_h\|_{L^2(\Omega)} = C\,\frac{h^{\mu+\nu-4}}{p^{r+s-4}}\sqrt{\sum_{K_i \in \mathcal{P}_h}\|u\|^2_{H^{r_i}(K_i)}}\;\sup_{q \in L^2(\Omega)}\frac{\sqrt{\sum_{K_i \in \mathcal{P}_h}\|w_q\|^2_{H^{s_i}(K_i)}}}{\|q\|_{L^2(\Omega)}}.$$

The term involving the supremum can be bounded in the same manner as previously done in the proof of Theorem 4, but we again have to determine the regularity of the functions w_q. In order to do so, we call upon Lemma 8 which states that for $\sigma = 0$ we need to find the roots $\lambda \in \mathbb{C}$ on the line $-2I$ of the following characteristic equations:

$$\sinh(\lambda\omega_j^i) = \pm\lambda\sin(\omega_j^i), \qquad j = 1, \ldots, N_c^i, \quad i = 1, \ldots, N.$$

By expanding $\lambda = a - 2I$, $a \in \mathbb{R}$, we get sets of equations that govern the real and imaginary parts of the above equations,

$$\begin{aligned}\sinh(a\omega_j^i)\cos(2\omega_j^i) &= \pm a\sin(\omega_j^i),\\ \cosh(a\omega_j^i)\sin(2\omega_j^i) &= \pm 2\sin(\omega_j^i).\end{aligned} \qquad (33)$$

The value $a = 0$ is never a root of these equations for $0 < \omega_j^i < \pi$ (no re-entrant corners), as substitution of $a = 0$ into $(33)^2$ gives:

$$\sin(2\omega_j^i) = \pm 2\sin(\omega_j^i) \quad \Rightarrow \quad \cos(\omega_j^i) = \pm 1 \quad \Rightarrow \quad \omega_j^i = 0, \pi.$$

So we can multiply $(33)^2$ by a, $(33)^1$ by 2, and subtract the resulting equations, which yields:

$$2\sinh(a\omega_j^i)\cos(2\omega_j^i) = \pm a \cosh(a\omega_j^i)\sin(2\omega_j^i), \quad a \neq 0,$$

which we can rewrite as follows:

$$\tanh(a\,\omega_j^i) = \frac{1}{2}\tan(2\omega_j^i)\,a, \quad a \neq 0.$$

Thus, the above equations are equivalent with (28), with $\lambda = a - 2I$. If there are no roots a other than $a = 0$, then according to Lemma 8 the functions w_q are in $H^4(\mathcal{P}_h)$. Thus, for $p = 2$, we obtain $\nu = 3$, and for $p \geq 3$, we get $\nu = 4$, which establishes the assertion. ∎

Corollary 6 *Let $\mathcal{P}_h$ be as defined in Theorem 5. In addition, assume that all corners of the polygons are equal to $\pi/2$. Then, for $p \geq 3$, the error $e_h = u - u_h$ converges optimally in $L^2(\Omega)$, i.e.*

$$\boxed{\|e_h\|_{L^2(\Omega)} \leq C\,\frac{h^\mu}{p^r}\sqrt{\sum_{K_i \in \mathcal{P}_h}\|u\|^2_{H^{r_i}(K_i)}}, \quad p \geq 3,\ r_i \geq 2,}$$

where $r = \min\limits_{K_i \in \mathcal{P}_h}\{r_i\}$, $p = \max\limits_{K_i \in \mathcal{P}_h}\{p_i\}$, $h = \max\limits_{K_i \in \mathcal{P}_h}\{h_i\}$, and $\mu = \min(p+1, r)$.

Proof: This is an immediate consequence of Theorem 5, as substitution of $\omega_j^i = \pi/2$ into (32) yields:

$$\tanh\left(\frac{a\pi}{2}\right) = 0,$$

which has only one root $a = 0$. ∎

From Theorem 5 we can conclude that in $L^2(\Omega)$ we always obtain suboptimal h convergence for $p = 2$. For $p \geq 3$, optimal rates can be obtained, but the corner angles have to satisfy the condition (32). This condition is satisfied for strictly rectangular meshes, but such meshes rarely occur in practical applications. Therefore, only in few occurrences will optimal convergence be guaranteed by Theorem 5.

4 Numerical Verifications

4.1 One Dimensional Tests

We consider a *one dimensional* version of problem (3):

$$-\frac{d^2u}{dx^2} + u = x + 1, \quad \text{for} \quad 0 < x < 1,$$

$$u(0) = 0, \ u(1) = 1.$$

The exact solution to this problem is:

$$u(x) = x + 1 - \frac{e^x + e^{1-x}}{1 + e}.$$

For $p = 2, 3, 4, 5$, solutions $\{u_h\}$ of (19) are computed by performing successive uniform h refinements. In Figure 4, the convergence results are shown for the norm $\|.\|_{H^2(\mathcal{P}_h)}$ (5), where the h convergence rates are computed according the following rule:

$$\rho_h = \frac{\log\left(e_h^i / e_h^{i+1}\right)}{\log 2}$$

The observed convergence rates in Figure 4 of order $p - 1$ confirm the rates that are predicted for convergence in $H^2(\mathcal{P}_h)$ (see Theorem 3). In Figures 5 and 6, the results are illustrated for the $H^1(\mathcal{P}_h)$ and $L^2(\Omega)$ norm, respectively. The h convergence rates in $H^1(\mathcal{P}_h)$ are of order p and agree with the prediction in Theorem 4. In $L^2(\Omega)$, the convergence rates are of order $p + 1$, for $p \geq 3$, and of order p for $p = 2$. These rates also confirm the rates predicted in Theorem 5 (for one-dimensional versions of (26), the dual solutions w_q are always in $H^4(\mathcal{P}_h)$ for $q(\cdot)$ that belong to $L^2(\Omega)$).
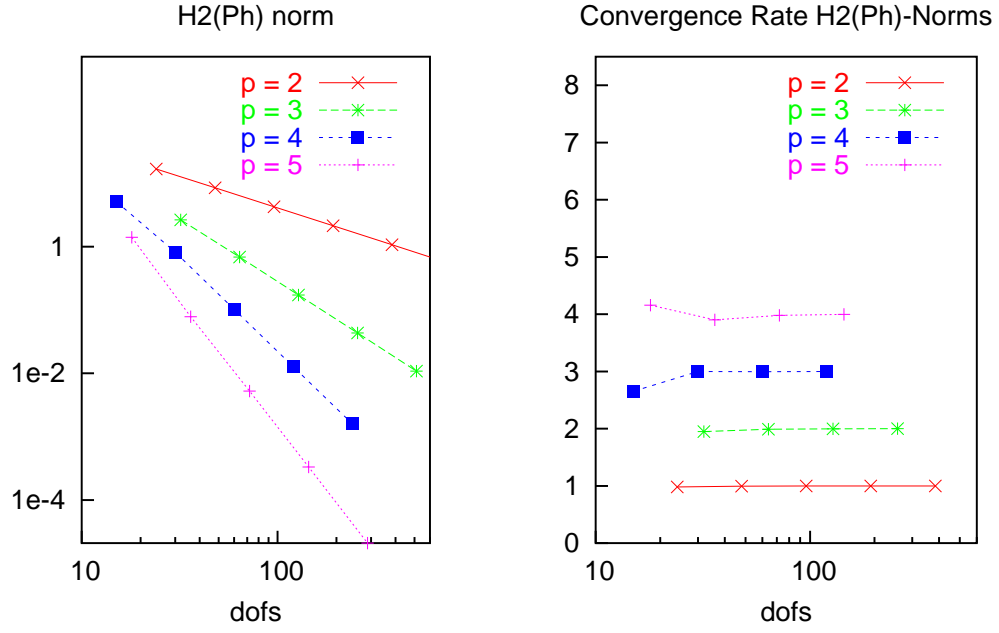
Fig. 4. Norm $\|\|\cdot\|\|$ of the approximation error (left) and uniform h convergence rates (right) versus number of degrees of freedom - 1D results.
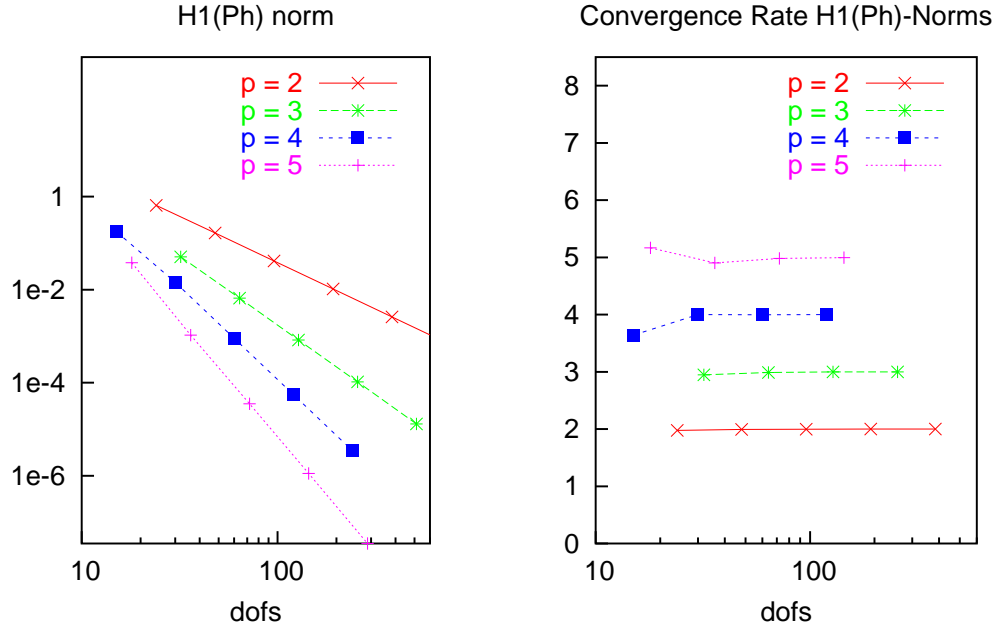

Fig. 5. $H^1(\mathcal{P}_h)$ norm of the approximation error (left) and uniform h convergence rates (right) versus number of degrees of freedom - 1D results.
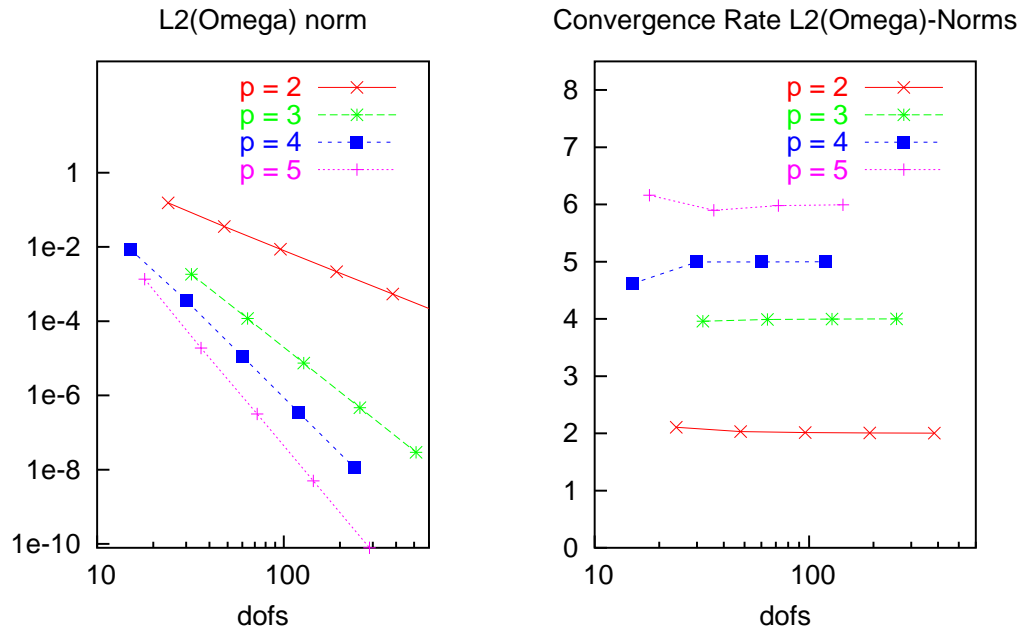
Fig. 6. $L^2(\Omega)$ norm of the approximation error (left) and uniform h convergence rates (right) versus number of degrees of freedom - 1D results.

4.2 Two Dimensional Tests

Next, we consider the two-dimensional test case on the unit square $\Omega = (0, 1) \times (0, 1)$:

$$-\Delta u + u = 0, \text{ in } \Omega,$$

$$u(x, y) = \begin{cases} 0, & \text{for } x = 0,\ 0 \le y \le 1, \\ 0, & \text{for } x = 1,\ 0 \le y \le 1, \\ 0, & \text{for } y = 0,\ 0 \le x \le 1, \\ \sin(2\pi x)\ \sinh(\sqrt{1 + 4\pi^2}), & \text{for } y = 1,\ 0 \le x \le 1, \end{cases}$$

The exact solution to this boundary value problem is:

$$u(x) = \sin(2\pi x)\ \sinh(\sqrt{1 + 4\pi^2}y) \tag{34}$$

For $p = 2, 3, 4, 5$, solutions $\{u_h\}$ of (19) are computed by performing successive uniform h refinements. In Figures 7 and 8, the convergence results are shown for the norms $\|.\|$ and $\|\cdot\|_{H^2(\mathcal{P}_h)}$, respectively. Note that the norm $\|\cdot\|$ represents a broken Laplacian norm on $\mathcal{P}_h$, whereas $\|\cdot\|_{H^2(\mathcal{P}_h)}$ uses the complete local Sobolev norm in H^2, *i.e.* it includes the local L^2 norm of the cross derivatives $\partial^2 u / \partial x \partial y$ (see also (5)). The results for the norm $\|\cdot\|_{H^2(\mathcal{P}_h)}$ confirm the predicted h convergence rates of Theorem 3. The convergence rates in the norm $\|\cdot\|$ are higher in the pre-asymptotic range but converge to the same rates as those observed for the norm $\|\cdot\|_{H^2(\mathcal{P}_h)}$.

Figure 9 shows the convergence of the error in the $H^1(\mathcal{P}_h)$ norm. The predicted rates, stated in Theorem 4, are confirmed by exhibiting convergence rates of order p.

Since the mesh consists of rectangular elements, Theorem 5 and Corollary 6 assert that the convergence rates in $L^2(\Omega)$ should be of order 2, for $p = 2$, and $p + 1$, for $p \ge 3$. The convergence results shown in Figure 5 agree with this assertion.

We consider an additional test problem on the quadrilateral domain depicted in Figure 11. The domain partitions $\{\mathcal{P}_h\}$ are performed as illustrated in this figure. Thus, the skewness of the elements is determined by the angle θ. The same Poisson equation as in the previous example is solved on the quadrilateral and the Dirichlet boundary conditions are applied such that the solution is as given in (34).

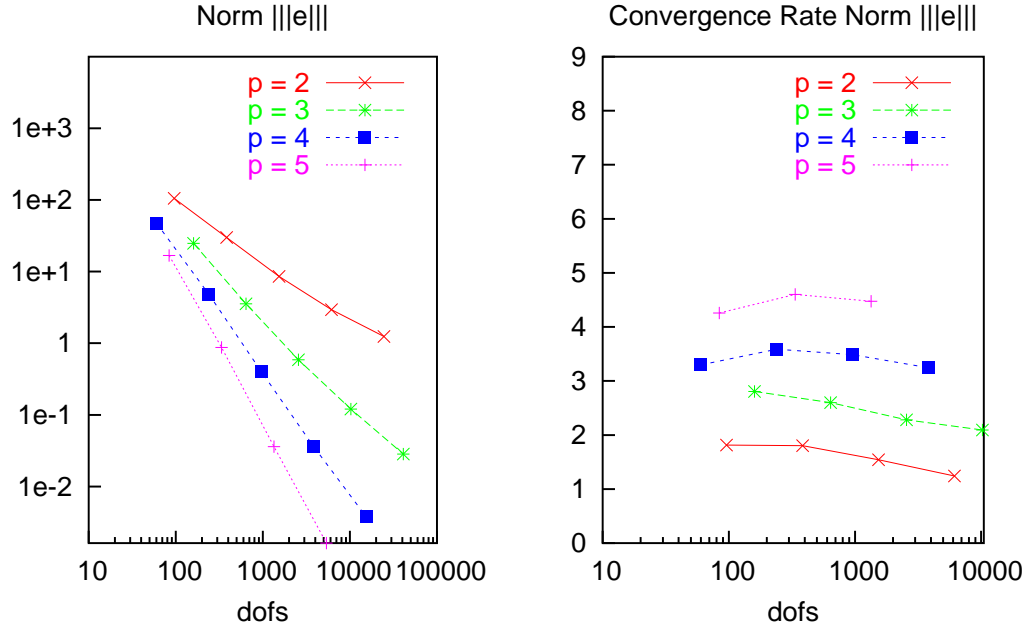In Figure 12, the uniform h convergence rates are presented for $\theta = \pi/6$. For

Fig. 7. Norm $\|\!\|\!\|\cdot\|\!\|\!\|$ of the approximation error (left) and uniform h convergence rates (right) versus number of degrees of freedom - 2D results.
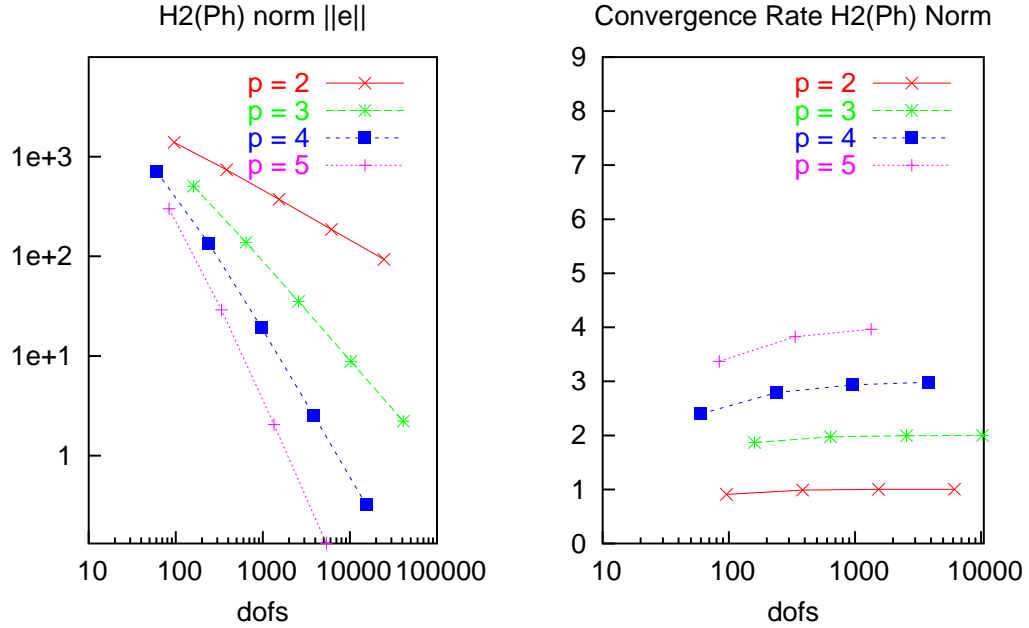

Fig. 8. $H^2(\mathcal{P}_h)$ norm of the approximation error (left) and uniform h convergence rates (right) versus number of degrees of freedom - 2D results.
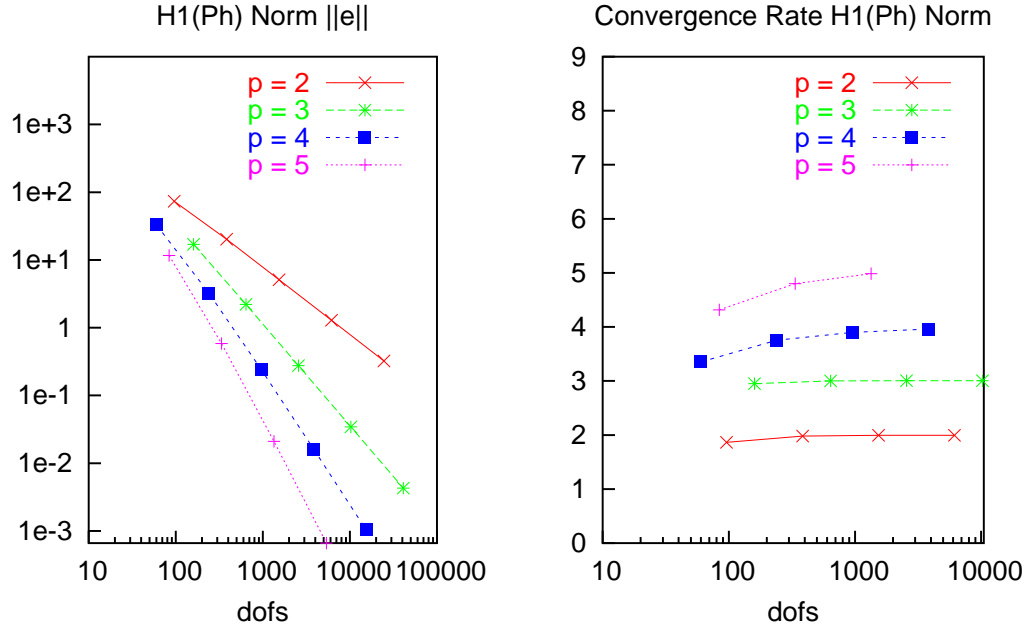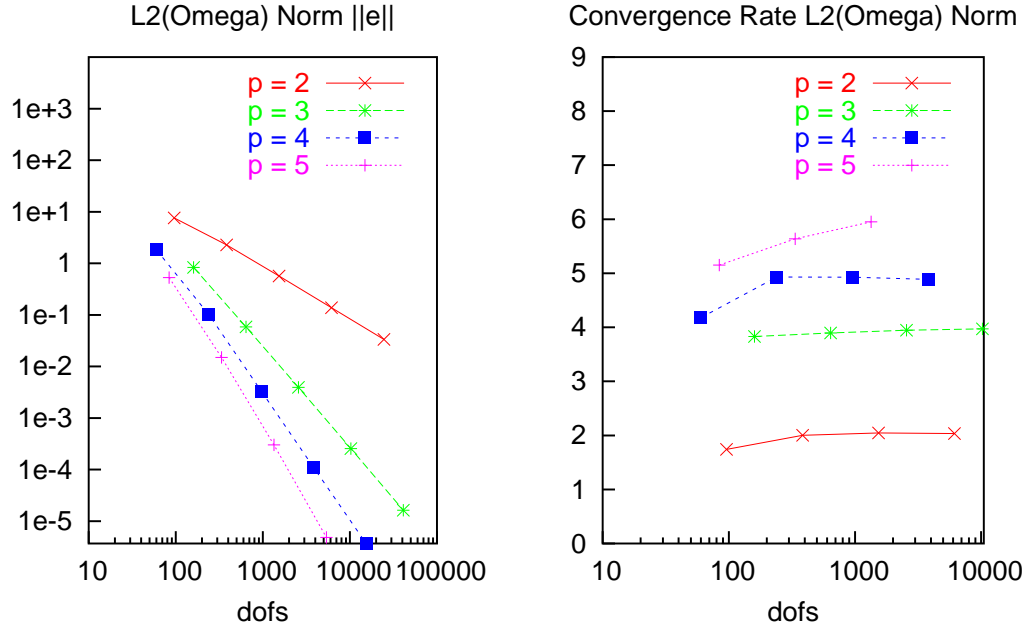
Fig. 9. $H^1(\mathcal{P}_h)$ norm of the approximation error (left) and uniform h convergence rates (right) versus number of degrees of freedom - 2D results.

Fig. 10. $L^2(\Omega)$ norm of the approximation error (left) and uniform h convergence rates (right) versus number of degrees of freedom - 2D results.

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convergence in $H^2(\mathcal{P}_h)$ and $H^1(\mathcal{P}_h)$, both figures show no noticeable difference with the results given in Figures 7 through 9. As expected, the skewness of the mesh does not affect the convergence of the error in these norms. However, the assertion of Theorem 5 for the convergence in the $L^2(\Omega)$ norm of skewed meshes, is only confirmed in part. The corner angles in the mesh are either $\pi/6$ or $5\pi/6$. For the latter of these, the characteristic equations (32) have nonzero roots. Thus, the convergence rates in $L^2(\Omega)$ should be suboptimal according Theorem 5. For the even order approximations, the suboptimal rate of order p is indeed observed, but for odd order approximation the optimal rate of $p+1$ is still obtained. This even-odd behavior in $L^2(\Omega)$ has also been observed for the DGM introduced by Oden, Babuška, and Baumann [4], but in their results the suboptimal $L^2(\Omega)$ convergence also emerges for rectangular meshes.

5   Concluding Remarks

A new DGM is introduced for the two-dimensional reaction-diffusion problem with prescribed Neumann and/or Dirichlet boundary conditions. The DGM formulation employs local second order derivatives, satisfies a local conservation property, and the corresponding bilinear form satisfies coercivity and continuity conditions on the broken Sobolev space $H^2(\mathcal{P}_h)$. Due to the coercivity property, the formulation is numerically stable and does not require any additional penalization.

We have derived *a priori* error estimates that show that optimal h *and* p convergence is obtained in $H^2(\mathcal{P}_h)$. If the mesh consists of convex polygons, then we can also prove optimal convergence in $H^1(\mathcal{P}_h)$. These assertions are confirmed by one- and two-dimensional experiments.
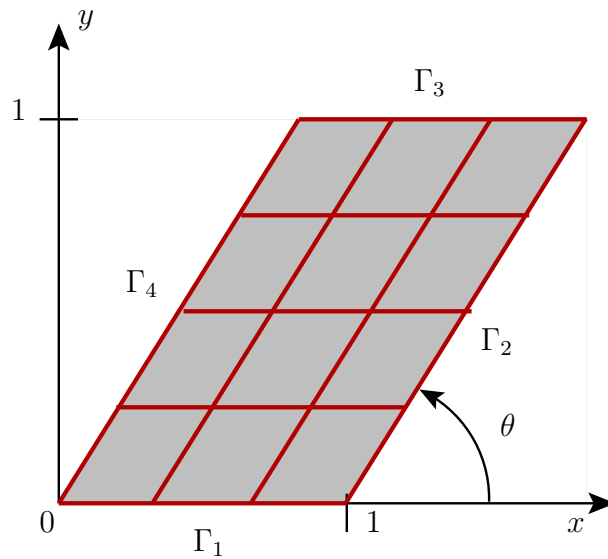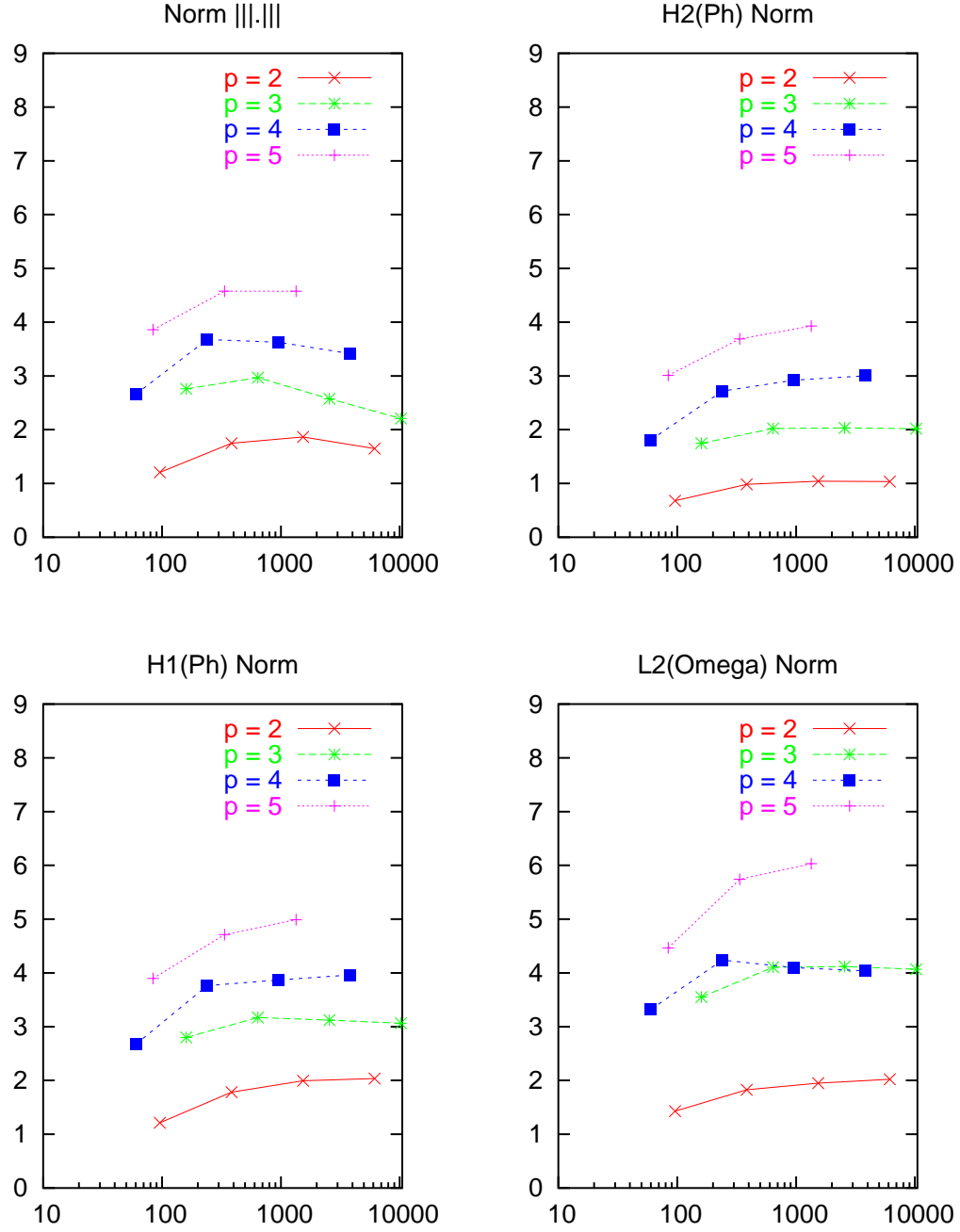

Fig. 11. Mesh geometries.

Fig. 12. Uniform h convergence rates versus number of degrees of freedom - 2D results - Mesh distortion of 30°.

In $L^2(\Omega)$, we always obtain a suboptimal h convergence rate of order p for $p = 2$. For $p \geq 3$, we can prove optimal h and p convergence if the corner angles of the polygons have no roots other than zero for the characteristic equation (32). If we employ strictly rectangular meshes, then this condition is satisfied and optimal convergence for $p \geq 3$ is obtained and confirmed by numerical results. In practical applications, meshes consist of elements with various shapes and the condition (32) is most likely not satisfied. In those cases, we can only prove suboptimal convergence rates. Remarkably, for odd order approximations on skewed meshes, numerical results show that optimal convergence is obtained, nevertheless. This 'odd-even' behavior of the error in $L^2(\Omega)$ is also observed in other DGM's (*e.g.* see [7]).

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